MAT 1332, Winter 2009, Assignment #3, Solutions

Total marks=29.

[2] 1. Using the substitution $u = 2+5x$, then $du = 5dx$, and then we find that $dx = \frac{du}{5}$, and the limits of integration are from $u = 2$ to $u = \infty$. Therefore,

$$\int_0^\infty \frac{1}{(2+5x)^4}dx = \int_2^\infty \frac{1}{5}u^{-4}dx = \frac{-1}{15}u^{-3}|_2^\infty = \frac{1}{120}.$$

[2] 2. This is the type II improper integral. Using the substitution $u = x - 1$, then $du = dx$, and the limits of integration are from $u = 0$ to $u = 2$. Therefore,

$$\int_1^3 \frac{5}{\sqrt{x-1}}dx = \int_0^2 \frac{5}{\sqrt{u}}du = \lim_{\epsilon\to 0^+}\int_\epsilon^2 \frac{5}{\sqrt{u}}du = \lim_{\epsilon\to 0^+} 5\cdot 2u^{\frac{1}{2}}|_\epsilon^2 = \lim_{\epsilon\to 0^+}(10\sqrt{2} - 10\epsilon^{\frac{1}{2}}) = 10\sqrt{2}.$$

[3] 3. For $x > 1$, $\sqrt{x} > 0$, then $\sqrt{x} + e^x > e^x$, so

$$\int_1^\infty \frac{1}{\sqrt{x}+e^x}dx < \int_1^\infty \frac{1}{e^x}dx = \lim_{T\to\infty}\int_1^T \frac{1}{e^x}dx = \lim_{T\to\infty}(-e^{-x}|_1^T) = \lim_{T\to\infty}(-e^{-T} + e^{-1}) = e^{-1}.$$

[2] So this integral is convergent.

[1] An upper bound is $e^{-1} = \frac{1}{e}$.

[4] 4. Let $b(t)$ denote the population of bacteria at time t. Then $b(0) = 1000$.

(a) [1] The pure-time differential equation is

$$\frac{db}{dt} = \frac{1000}{(2+3t)^{1.5}}.$$

(b) [2]

Approach 1: The population starting with $b(0) = 1000$, the total change between $t = 0$ and t will be given by the definite integral $\int_0^t \frac{1000}{(2+3s)^{1.5}}ds$, so we can write down the solution for this equation as follows,

$$\begin{aligned}
b(t) &= 1000 + \int_0^t \frac{1000}{(2+3s)^{1.5}}ds \\
&\quad (\text{using the substitution}\, u = 2+3s, \text{ then}\, ds = \frac{du}{3}, \\
&\quad \text{the limits of integration are from}\, u = 2\, \text{to}\, u = 2+3t.) \\
&= 1000 + \int_2^{2+3t} \frac{1000}{3}\cdot\frac{1}{u^{1.5}}du \\
&= 1000 + \frac{1000}{3}\int_2^{2+3t} \cdot\frac{1}{u^{1.5}}du \\
&= 1000 + \frac{1000}{3}\cdot(-\frac{1}{0.5}u^{-0.5}|_2^{2+3t}) \\
&= 1000 + \frac{2000}{3}\Big(-u^{-0.5}|_2^{2+3t}\Big) \\
&= 1000 + \frac{2000}{3}[-(2+3t)^{-0.5} + 2^{-0.5}] \\
&= 1000 + \frac{2000}{3}\cdot 2^{-0.5} - \frac{2000}{3}(2+3t)^{-0.5} \\
&\approx 1471 - 667\cdot(2+3t)^{-0.5}.
\end{aligned}$$

Since the population would approach a limiting size of 1471, the rule of this sort of growth can be maintained indefinitely.

Approach 2: To answer this question, we can just need to test the convergence of the improper integral

$$\int_0^\infty \frac{1000}{(2+3t)^{1.5}}dt.$$

For this propose, by the definition of the type I improper integral and the substitution $u = 2+3t$ again, we have

$$\begin{aligned}
\int_0^\infty \frac{1000}{(2+3t)^{1.5}}dt &= \lim_{T\to\infty}\int_0^T \frac{1000}{(2+3t)^{1.5}}dt = \lim_{T\to\infty}\int_2^{2+3T} \frac{1000}{3}\cdot\frac{1}{u^{1.5}}du \\
&= \frac{1000}{3}\lim_{T\to\infty}\int_2^{2+3T}\frac{1}{u^{1.5}}du = \frac{1000}{3}\lim_{T\to\infty}\Big(-\frac{1}{0.5}u^{-0.5}|_2^{2+3T}\Big) \\
&= \frac{2000}{3}\lim_{T\to\infty}\Big(-u^{-0.5}|_2^{2+3T}\Big) \\
&= \frac{2000}{3}\lim_{T\to\infty}[-(2+3T)^{-0.5}+2^{-0.5}] \\
&= \frac{2000}{3}\cdot 2^{-0.5} \approx 471,
\end{aligned}$$

which is a number. This implies that the improper integral

$$\int_0^\infty \frac{1000}{(2+3t)^{1.5}}dt$$

is convergent. Therefore we can conclude that the rule of this sort of growth can be maintained indefinitely.

(c) [1] The population will not ever reach 2000 since the limiting size is $1471 < 2000$ as t approaches infinity. (From Approach 2, we can also find out the limiting size is $1000+471 = 1471$ approximately. So we can draw the same conclusion for part (c).)

[4] 5. Noting that this nonautonomous differential equation is separable. For

$$\frac{db}{dt} = e^{-t}b.$$

To separate variables,

$$\frac{db}{b} = e^{-t}dt.$$

To integrate both sides,

$$\begin{aligned}
\int\frac{db}{b} &= \int e^{-t}dt, \\
\ln|b| + C_1 &= -e^{-t}+C_2. \quad (C_1, C_2 \text{ are the integration constants.})
\end{aligned}$$

Combing the constants,

$$\ln|b| = -e^{-t}+C.$$

Solving for b,

$$|b| = e^{-e^{-t}+C}.$$

Rewriting the constant as $K = e^C$,

$$|b| = Ke^{-e^{-t}}.$$

So we can have two possible solutions $b = Ke^{-e^{-t}}$ and $b = -Ke^{-e^{-t}}$. Now applying the initial condition $b(0)$ for each solution, we have

For $b = Ke^{-e^{-t}}$, $10^6 = b(0) = Ke^{-e^{-0}}$, we have $K = 10^6 e$. Then the solution is

$$b(t) = 10^6 e \cdot e^{-e^{-t}} = 10^6 e^{-e^{-t}+1}.$$

For $b = Ke^{-e^{-t}}$, $10^6 = b(0) = -Ke^{-e^{-0}}$, we have $K = -10^6 e$. Then the solution is

$$b(t) = -(-10^6 e \cdot e^{-e^{-t}}) = 10^6 e^{-e^{-t}+1}.$$

So in the end, you will see we only have one solution

$$b(t) = 10^6 e^{-e^{-t}+1}.$$

[6] 6. The differential equations are autonomous.

(a) [3] To solve the equation by using the method of separation of variables.

$$\begin{aligned} \frac{dV}{2\sqrt{V}} &= -dt && \text{(separating variables)} \\ \sqrt{V} + C_1 &= -t + C_2 && \text{(integrating)} \\ \sqrt{V} &= -t + C && \text{(combing constants)} \\ V &= (-t + C)^2. && \text{(solving for } y\text{)} \end{aligned}$$

Applying the initial condition $V(0) = 16$, we have $16 = V(0) = C^2$, or $C = 4$. Therefore, the solution is

$$V(t) = (4 - t)^2.$$

(b) [1] To solve $0 = (4 - t)^2$, we know that the tank is empty at $t = 4$. See the graph of the solution $V(t) = (4 - t)^2$ in Figure 1.

(c) [2] To solve $\frac{dW}{dt} = -2W$ with the initial condition $W(0) = 16$ by using the method of separation of variables.

$$\begin{aligned} \frac{dW}{W} &= -2dt && \text{(separating variables)} \\ \ln|W| + C_1 &= -2t + C_2 && \text{(integrating)} \\ \ln|W| &= -2t + C && \text{(combing constants)} \\ |W| &= e^{-2t+C} && \text{(solving for } y\text{)} \\ |W| &= Ke^{-2t}. && \text{(rewriting the constant as } K = e^C\text{)} \end{aligned}$$

Here, similar to Question #5, there will be two possible solutions depending on the sign of W. But we do not need to take care of this now. As you have already seen in Question #5, these two solutions will turn out be one after the initial condition are applied. So we just write down

$$W = Ke^{-2t}.$$

Applying the initial condition $W(0) = 16$, we have $16 = W(0) = Ke^{-2\cdot 0}$, or $K = 16$. Therefore, the solution is

$$W(t) = 16e^{-2t}.$$

From part (b), we know that the tank which follow Torricelli's law must stay at zero since the tank will be empty when $t = 4$. But if it followed the law of exponential decay, the depth would be $16e^{-2\cdot 4} = 16e^{-8} \approx 1.8 \times 10^{-6}$, which is very low, but not actually zero.

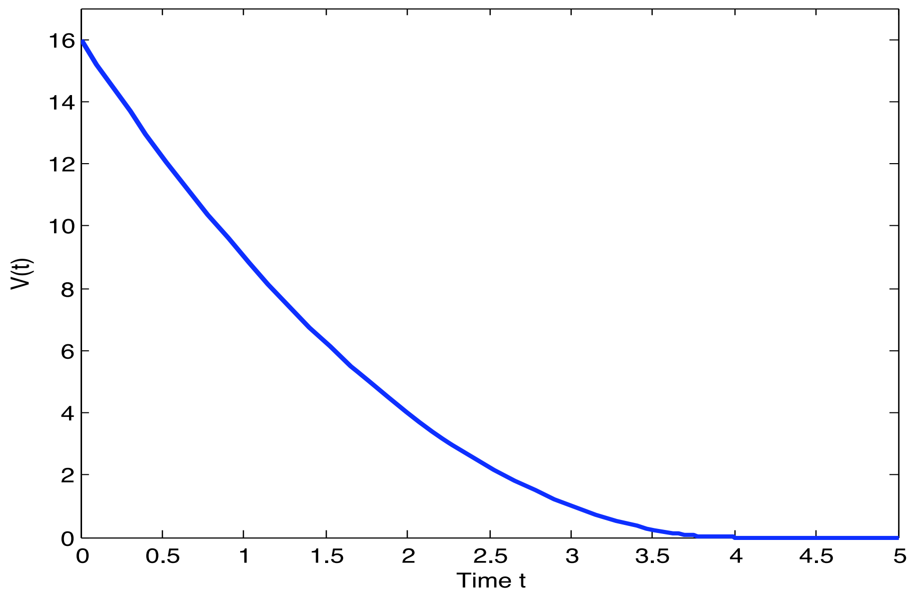

Figure 1: The solution $V(t) = (4-t)^2$ for $t \in (0,5)$.

[8] 7. Noting that this differential equation is autonomous.

(a) [2] To find the steady state, we write down the equation for the equilibria,

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ (x-2)(x-3) &= 0 \\ x - 2 = 0 \quad \text{or} \quad & x - 3 = 0 \quad \text{(set each factor equal to 0)} \\ x = 2 \quad \text{or} \quad & x = 3 \quad \text{(solve each equation)} \end{aligned}$$

So the steady states $x_1^* = 2$, $x_2^* = 3$.

(b) [2] The derivative of the rate of change $f(x)$ with respect to x is

$$f'(x) = 2x - 5.$$

Then

$$\begin{aligned} f'(x_1^*) &= f'(2) = 2 \cdot 2 - 5 = -1 < 0, \\ f'(x_2^*) &= f'(3) = 2 \cdot 3 - 5 = 1 > 0, \end{aligned}$$

hence, $x_1^* = 2$ is stable, and $x_2^* = 3$ is unstable.

(c) [1] The phase line diagram is shown in Figure 2.

(d) [3] Noting that this autonomous differential equation is separable.
For

$$\frac{dx}{dt} = x^2 - 5x + 6.$$

To separate variables,

$$\frac{dx}{x^2 - 5x + 6} = dt.$$

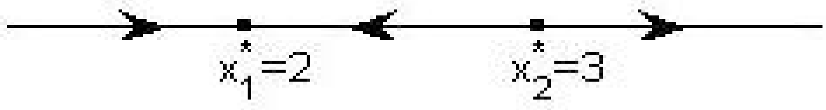

Figure 2: The phase line diagram for $\frac{dx}{dt} = x^2 - 5x + 6$.

To integrate both sides,

$$\begin{aligned}\int \frac{dx}{x^2-5x+6} &= \int dt,\\ \int \frac{1}{(x-2)(x-3)}dx &= \int dt.\end{aligned}$$

For the left-hand side of equation, using partial fraction, we know that

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)} = \frac{(A+B)x-(3A+2B)}{(x-2)(x-3)}.$$

Then we have $A+B=0$, $-(3A+2B)=1$. Solve these, we find out $A=-1, B=1$. Thus,

$$\int \frac{1}{(x-2)(x-3)}dx = \int \frac{-1}{(x-2)} + \frac{1}{(x-3)}dx.$$

Therefore,

$$\begin{aligned}\int \frac{-1}{(x-2)} + \frac{1}{(x-3)}dx &= \int dt,\\ -\ln|x-2| + \ln|x-3| + C_1 &= t + C_2. \quad (C_1, C_2 \text{ are the integration constants.})\end{aligned}$$

Combing the constants,

$$-\ln|x-2| + \ln|x-3| = t + C.$$

Solving for x,

$$\begin{aligned}\ln\left|\frac{x-3}{x-2}\right| &= t + C,\\ \left|\frac{x-3}{x-2}\right| &= e^{t+C}.\end{aligned}$$

Rewriting the constant as $K = e^C$,

$$\left|\frac{x-3}{x-2}\right| = Ke^t.$$

Here, similar to Question #5, there will be two possible solutions depending on the sign of $\frac{x-3}{x-2}$. But we do not need to take care of this now. As you have already seen in Question #5, these two solutions will turn out be one after the initial condition are applied. So we just write down

$$\begin{aligned}\frac{x-3}{x-2} &= Ke^t\\ x-3 &= (x-2)Ke^t\\ x-3 &= Ke^t x - 2Ke^t\\ x(1-Ke^t) &= 3-2Ke^t\\ x &= \frac{3-2Ke^t}{1-Ke^t}.\end{aligned}$$

Now applying the initial condition $x(0) = 0$, we have

$$0 = x(0) \quad = \quad \frac{3 - 2Ke^0}{1 - Ke^0},$$

then we have $K = \frac{3}{2}$. Therefore, the solution is

$$x(t) = \frac{3 - 2 \cdot \frac{3}{2} \cdot e^t}{1 - \frac{3}{2} \cdot e^t} = \frac{3 - 3e^t}{1 - \frac{3}{2}e^t} = \frac{6(1 - e^t)}{2 - 3e^t}.$$